The centralisers of nilpotent elements in the classical Lie algebras

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Introduction

Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{K}$. Consider the coadjoint representation $\mathrm{ad}^*(\mathfrak{g})$. *The index of* $\mathfrak{g}$ is the minimum of dimensions of stabilisers $\mathfrak{g}_\alpha$ over all covectors $\alpha \in \mathfrak{g}^*$

$$\operatorname{ind}\mathfrak{g} = \min_{\alpha\in\mathfrak{g}^*} \dim \mathfrak{g}_\alpha.$$

The definition of index goes back to Dixmier [3, 11.1.6]. This notion is important in Representation Theory and also in Invariant Theory. By Rosenlicht's theorem [12], generic orbits of an arbitrary action of a linear algebraic group on an irreducible algebraic variety are separated by rational invariants; in particular, $\operatorname{ind}\mathfrak{g} = \mathrm{tr.deg}\,\mathbb{K}(\mathfrak{g}^*)^G$.

The index of a reductive algebra equals its rank. Computing the index of an arbitrary Lie algebra seems to be a wild problem. However, there is a number of interesting results for several classes of non-reductive subalgebras of reductive Lie algerbas. For instance, parabolic subalgebras and their "relatives" (nilpotent radicals, seaweeds) are considered in [4], [8], [13]. The centralisers of elements form another interesting class of subalgebras. The last topic is closely related to the theory of integrable Hamiltonian systems.

Let G be a semisimple Lie group (complex or real), $\mathfrak{g} = \operatorname{Lie} G$, and Gx an orbit of a covector $x \in \mathfrak{g}^*$. Let $\mathfrak{g}_x$ denote the stabiliser of x. It is well-known that the orbit Gx possesses a G-invariant symplectic structure. There is a family of commuting with respect to a Poisson bracket polynomial functions on $\mathfrak{g}^*$ constructed by the argument shift method such that its restriction to Gx contains $\frac{1}{2}\dim(Gx)$ algebraically independent functions if and only if $\operatorname{ind}\mathfrak{g}_x = \operatorname{ind}\mathfrak{g}$, [1].

Conjecture (Élashvili). *Let $\mathfrak{g}$ be a reductive Lie algebra. Then $\operatorname{ind}\mathfrak{g}_x = \operatorname{ind}\mathfrak{g}$ for each covector $x \in \mathfrak{g}^*$.*

Recall that if $\mathfrak{g}$ is reductive, then the $\mathfrak{g}$-modules $\mathfrak{g}^*$ and $\mathfrak{g}$ are isomorphic. In particular, it is enough to prove the "index conjecture" for stabilisers of vectors $x \in \mathfrak{g}$.

Given $x \in \mathfrak{g}$, let $x = x_s + x_n$ be the Jordan decomposition. Then $\mathfrak{g}_x = (\mathfrak{g}_{x_s})_{x_n}$. The subalgebra $\mathfrak{g}_{x_s}$ is reductive and contains a Cartan subalgebra of $\mathfrak{g}$. Hence, $\operatorname{ind}\mathfrak{g}_{x_s} = \operatorname{ind}\mathfrak{g} = \operatorname{rk}\mathfrak{g}$. Thus, a verification of the "index conjecture" is reduced to the computation of $\operatorname{ind}\mathfrak{g}_{x_n}$ for nilpotent elements $x_n \in \mathfrak{g}$. Clearly, we can restrict ourselves to the case of simple $\mathfrak{g}$.

Note that if x is a regular element, then the stabiliser $\mathfrak{g}_x$ is commutative and of dimension $\operatorname{rk}\mathfrak{g}$. The "index conjecture" was proved for subregular nilpotents and nilpotents of height 2 [9], and also for nilpotents of height 3 [10]. (The height of a nilpotent element e is the maximal number m such that $(\operatorname{ad} e)^m \neq 0$.) Recently, Élashvili's conjecture was proved by Charbonnel [2] for $\mathbb{K} = \mathbb{C}$.

In the present article, we prove in an elementary way, that for any nilpotent element $e \in \mathfrak{g}$ of a simple classical Lie algebra the index of $\mathfrak{g}_e$ equals the rank of $\mathfrak{g}$. We assume that the

ground field $\mathbb{K}$ contains at least k elements, where k is the number of Jordan blocks of a nilpotent element $e \in \mathfrak{g}$. For the orthogonal and symplectic algebras, it is also assumed that $\operatorname{char}\mathbb{K} \neq 2$. Note that if a reductive Lie algebra $\mathfrak{g}$ does not contain exceptional ideals, then $\mathfrak{g}_{x_s}$ has the same property. Thus, the "index conjecture" is proved for the direct sums of classical algebras.

By Vinberg's inequality, which is presented in [9, Sect. 1], we have $\operatorname{ind}\mathfrak{g}_x \geq \operatorname{ind}\mathfrak{g}$ for each element $x \in \mathfrak{g}^*$. It remains to prove the opposite inequality. To this end, it suffices to find $\alpha \in (\mathfrak{g}_x)^*$ such that the dimension of its stabiliser in $\mathfrak{g}_x$ is at most $\operatorname{rk}\mathfrak{g}$. For $\mathfrak{g} = \mathfrak{gl}(V)$ and $\mathfrak{g} = \mathfrak{sp}(V)$, we explicitly indicate such a point $\alpha \in \mathfrak{g}_e^*$. In case of the orthogonal algebra, the proof is partially based on induction.

In the last two sections, $\mathbb{K}$ is assumed to be algebraically closed and of characteristic zero. It is shown that the stabilisers $(\mathfrak{g}_e)_\alpha$ constructed for $\mathfrak{g} = \mathfrak{gl}(V)$ and $\mathfrak{g} = \mathfrak{sp}(V)$ are generic stabilisers for the coadjoint representation of $\mathfrak{g}_e$. For the orthogonal case, we give an example of a nilpotent element $e \in \mathfrak{so}_8$ such that the coadjoint action of $\mathfrak{g}_e$ has no generic stabiliser. Similar results for parabolic and seaweed subalgebras of simple Lie algebras were obtained by Panyushev and also by Tauvel and Yu. In [13] there is an example of a parabolic subalgebra of $\mathfrak{so}_8$ having no generic stabilisers for the coadjoint representation. The affirmative answer for series A and C is obtained by Panyushev in [11].

In the last section, we consider the commuting variety of $\mathfrak{g}_e$ and its relationship with the commuting variety of triples of matrices.

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1. Preliminaries

Suppose $\mathfrak{g}$ is a simple classical Lie algebra or a general linear algebra. Let $e \in \mathfrak{g}$ be a nilpotent element and $\mathfrak{z}(e)$ its centraliser in $\mathfrak{g}$. Note that there is no essential difference between $\mathfrak{g} = \mathfrak{gl}(V)$ and $\mathfrak{g} = \mathfrak{sl}(V)$. However, the first case is more suitable for calculations. In case of orthogonal and symplectic algebras, we need some facts from the theory of symmetric spaces.

Let $(\ ,\ )_V$ be a non-degenerate symmetric or skew-symmetric form on a finite dimensional vector space V given by a matrix J, i.e., $(v, w)_V = v^t J w$, where the symbol ^{t} stands for the transpose. The elements of $\mathfrak{gl}(V)$ preserving $(\ ,\ )_V$ are exactly the fixed vectors $\mathfrak{gl}(V)^\sigma$ of the involution $\sigma(\xi) = -J\xi^t J^{-1}$. There is the $\mathfrak{gl}(V)^\sigma$-invariant decomposition $\mathfrak{gl}(V) = \mathfrak{gl}(V)^\sigma \oplus \mathfrak{g}_1$. The elements of $\mathfrak{g}_1$ multiply the form $(\ ,\ )_V$ by -1, i.e., $(\xi v, w)_V = (v, \xi w)_V$ for every $v, w \in V$.

Set $\mathfrak{g} = \mathfrak{gl}(V)^\sigma$, and let $e \in \mathfrak{g}$ be a nilpotent element. Denote by $\mathfrak{z}(e)$ and $\mathfrak{z}_{\mathfrak{gl}}(e)$ the centralisers of e in $\mathfrak{g}$ and $\mathfrak{gl}(V)$, respectively. Since $\sigma(e) = e$, σ acts on $\mathfrak{z}_{\mathfrak{gl}}(e)$. Clearly, $\mathfrak{z}_{\mathfrak{gl}}(e)^\sigma = \mathfrak{z}(e)$. This yields the $\mathfrak{z}(e)$-invariant decomposition $\mathfrak{z}_{\mathfrak{gl}}(e) = \mathfrak{z}(e) \oplus \mathfrak{z}_1$. Given $\alpha \in \mathfrak{z}_{\mathfrak{gl}}(e)^*$, let $\tilde{\alpha}$ denote its restriction to $\mathfrak{z}(e)$.

Proposition 1. *Suppose $\alpha \in \mathfrak{z}_{\mathfrak{gl}}(e)^*$ and $\alpha(\mathfrak{z}_1) = 0$. Then $\mathfrak{z}(e)_{\tilde{\alpha}} = \mathfrak{z}_{\mathfrak{gl}}(e)_\alpha \cap \mathfrak{z}(e)$.*

Proof. Take $\xi \in \mathfrak{z}(e)$. Since $[\xi, \mathfrak{z}_1] \subset \mathfrak{z}_1$, $\alpha([\xi, \mathfrak{z}(e)]) = 0$ if and only if $\alpha([\xi, \mathfrak{z}_{\mathfrak{gl}}(e)]) = 0$. In particular, $\mathfrak{z}(e)_{\tilde{\alpha}} = \mathfrak{z}(e)_\alpha$. $\square$

Suppose $\mathfrak{h}$ is a Lie algebra and $\tau \in \operatorname{Aut}\mathfrak{h}$ an involution, which defines the decomposition $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$. Each point $\gamma \in \mathfrak{h}_0^*$ determines a skew-symmetric 2-form $\hat{\gamma}$ on $\mathfrak{h}_1$ by $\hat{\gamma}(\xi, \eta) = \gamma([\xi, \eta])$.

Lemma 1. *In the above notation, we have* $\operatorname{ind}\mathfrak{h} \le \operatorname{ind}\mathfrak{h}_0 + \min\limits_{\gamma \in \mathfrak{h}_0^*} \dim(\operatorname{Ker}\hat{\gamma})$.

Proof. Consider γ as a function on $\mathfrak{h}$, which is equal to zero on $\mathfrak{h}_1$. Then $\mathfrak{h}_\gamma = (\mathfrak{h}_0)_\gamma \oplus (\mathfrak{h}_\gamma \cap \mathfrak{h}_1) = (\mathfrak{h}_0)_\gamma \oplus (\operatorname{Ker}\hat{\gamma})$. We have $\dim(\mathfrak{h}_0)_\gamma = \operatorname{ind}\mathfrak{h}_0$ for generic points (= points of some Zariski open subset $U_1 \subset \mathfrak{h}_0^*$). The points of $\mathfrak{h}_0^*$, where $\operatorname{Ker}\hat{\gamma}$ has the minimal possible dimension, form another open subset, say $U_2 \subset \mathfrak{h}_0^*$. For the points of the intersection $U_1 \cap U_2$, the dimension of the stabiliser in $\mathfrak{h}$ equals the required sum. $\square$

2. General linear algebra

Consider a nilpotent element $e \in \mathfrak{gl}(V)$, where V is an n-dimensional vector space over $\mathbb{K}$. Denote by $\mathfrak{z}(e)$ the centraliser of e. Let us show that the index of $\mathfrak{z}(e)$ equals n.

Let k be a number of Jordan blocks of e and $W \subset V$ a k-dimensional complement of $\operatorname{Im} e$ in V. Denote by $d_i + 1$ the dimension of i-th Jordan block. Choose a basis $w_1, w_2, \ldots, w_k$ in W, where w_i is a generator of an i-th Jordan block, i.e., the vectors $e^{s_i} w_i$ with $1 \le i \le k$, $0 \le s_i \le d_i$ form a basis of V. Let $\varphi \in \mathfrak{z}(e)$. Since $\varphi(e^s w_i) = e^s \varphi(w_i)$, the map φ is completely determined by its values on w_i, $i = 1, \ldots, k$. Each value $\varphi(w_i)$ can be written as

$$\varphi(w_i) = \sum_{j,s} c_i^{j,s} e^s w_j, \text{ where } c_i^{j,s} \in \mathbb{K}. \tag{1}$$

That is, φ is completely determined by the coefficients $c_i^{j,s} = c_i^{j,s}(\varphi)$. Note that $\varphi \in \mathfrak{z}(e)$ preserves the space of each Jordan block if and only if $c_i^{j,s}(\varphi) = 0$ for $i \ne j$.

The centraliser $\mathfrak{z}(e)$ has a basis $\{\xi_i^{j,s}\}$, where

$$\begin{cases} \xi_i^{j,s}(w_i) = e^s w_j, \\ \xi_i^{j,s}(w_t) = 0 \text{ for } t \ne i, \end{cases} \qquad \left[\begin{array}{l} d_j - d_i \le s \le d_j \text{ for } d_j \ge d_i, \\ 0 \le s \le d_i \text{ for } d_j \end{array} \right.$$

Consider a point $\alpha \in \mathfrak{z}(e)^*$ defined by the formula

$$\alpha(\varphi) = \sum_{i=1}^{k} a_i \cdot c_i^{i,d_i}, \;\; a_i \in \mathbb{K},$$

where $c_i^{j,s}$ are the coefficients of $\varphi \in \mathfrak{z}(e)$ and $\{a_i\}$ are non-zero pairwise distinct numbers. We have $\alpha(\xi_i^{j,s}) = a_i$ if $i = j$, $s = d_i$ and zero otherwise.

Theorem 1. *The stabiliser $\mathfrak{z}(e)_\alpha$ of α in $\mathfrak{z}(e)$ consist of all maps preserving the Jordan blocks, i.e., $\mathfrak{z}(e)_\alpha$ is the linear span of the vectors $\xi_i^{i,s}$.*

Proof. Suppose $\varphi \in \mathfrak{z}(e)$ is defined by formula (1). (Some of $c_i^{j,s}$ have to be zeros, but this is immaterial here). For each basis vector $\xi_i^{j,b}$, we have

$$\alpha([\varphi, \xi_i^{j,b}]) = \alpha(\sum_{t,s} c_t^{i,s} \xi_t^{j,s+b} - \sum_{t,s} c_j^{t,s} \xi_i^{t,s+b}) = a_j \cdot c_j^{i,d_j-b} - a_i \cdot c_j^{i,d_i-b} .$$

The element φ lies in $\mathfrak{z}(e)_\alpha$ if and only if $\alpha([\varphi, \xi_i^{j,b}]) = 0$ for all $\xi_i^{j,b}$.

Note that if φ preserves the Jordan blocks, i.e., $c_i^{j,s} = 0$ for $i \neq j$, then $\alpha([\varphi, \mathfrak{z}(e)] = 0$. Let us show that $\mathfrak{z}(e)_\alpha$ contains no other elements. Assume that $c_i^{j,s} \neq 0$ for some $i \neq j$. We have three different possibilities: $d_i < d_j$, $d_i = d_j$ and $d_i > d_j$.

If $d_j \leq d_i$, then put $\xi(w_j) = e^{(d_i-s)} w_i$ and $\xi(w_t) = 0$ for $t \neq j$. It should be noted that $0 \leq s \leq d_j \leq d_i$, hence the expression $e^{(d_i-s)}$ is well defined. One has to check that $e^{d_j+1}(\xi(w_j)) = 0$. Adding the powers of e, we get $e^{d_j+1}(\xi(w_j)) = e^{d_j+1+d_i-s} w_i = e^{d_j-s}(e^{d_i+1} w_i) = 0$. We have $\alpha([\varphi, \xi]) = a_i \cdot c_i^{j,s} - a_j \cdot c_i^{j,d_j-d_i+s}$. In case $d_j = d_i$, we obtain $(a_i - a_j) \cdot c_i^{j,s} \neq 0$. If $d_j > d_i$, then $s > d_j - d_i + s$. Choose the minimal s such that $c_i^{j,s} \neq 0$. For this choice, we get $\alpha([\varphi, \xi]) = a_i \cdot c_i^{j,s} \neq 0$.

Suppose now that $d_j > d_i$ and s is the minimal number such that $c_i^{j,s} \neq 0$. Set $\xi(w_j) = e^{(d_j-s) w_i}$ and $\xi(w_t) = 0$ for $t \neq j$. As in the previous case, we have $0 \leq s \leq d_j$. In particular, $d_j - s \geq 0$, $(d_j + 1 + d_j - s) > d_i + 1$ and, thereby, $e^{d_j+1}(\xi(w_j)) = 0$. We obtain

$$\alpha([\xi, \varphi]) = a_j \cdot c_i^{j,s} - a_i \cdot c_i^{j,d_i-d_j+s} = a_j \cdot c_i^{j,s} \neq 0.$$

Here $c_i^{j,d_i-d_j+s} = 0$, since $d_i - d_j + s < s$. $\square$

Corollary. *The index of $\mathfrak{z}(e)$ equals n.*

Proof. The stabiliser $\mathfrak{z}(e)_\alpha$ consist of all maps preserving Jordan blocks. In particular, it has dimension n. Hence, $\operatorname{ind} \mathfrak{z}(e) \leq n$. On the other hand, it follows from Vinberg's inequality that $\operatorname{ind} \mathfrak{z}(e) \geq n = \operatorname{rk} \mathfrak{gl}(V)$. $\square$

Let us give another proof of the inequality $\operatorname{ind} \mathfrak{z}(e) \leq n$.

Example 1. Let $e \in \mathfrak{gl}_n$ be a nilpotent element and $\mathfrak{h} = \mathfrak{z}(e)$ the centraliser of e. We may assume that the first Jordan block of e is of maximal dimension. Then $V = V_{d_1+1} \oplus V_{\rm oth}$ and $e = e_1 + e_2$, where V_{d_1+1} is the space of the first Jordan block and $V_{\rm oth}$ is the space of all other Jordan blocks; $e_1 \in \mathfrak{gl}_{d_1+1}$, $e_2 \in \mathfrak{gl}_{n-d_1-1}$. Let $\tau \in \mathfrak{gl}(V)$ be the conjugation by a diagonal matrix of order two such that $\mathfrak{gl}(V)^\tau = \mathfrak{gl}_{d_1+1} \oplus \mathfrak{gl}_{n-d_1-1}$. The involution τ acts on $\mathfrak{h} = \mathfrak{z}(e)$ and induces the decomposition $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$, where $\mathfrak{h}_0 = \mathfrak{z}(e_1) \oplus \mathfrak{z}(e_2)$ (the centralisers are considered in the algebras $\mathfrak{gl}_{d_1+1}$ and $\mathfrak{gl}_{n-d_1-1}$, respectively). Assume that the "index conjecture" is true for all $m < n$; in particular, $\operatorname{ind} \mathfrak{z}(e_2) = n - d_1 - 1$. The subalgebra $\mathfrak{z}(e_1)$ is commutative and its index equals $d_1 + 1$. According to Lemma 1, $\operatorname{ind} \mathfrak{z}(e) \leq \operatorname{ind}(\mathfrak{z}(e_1) \oplus \mathfrak{z}(e_2)) + \min_{\gamma \in \mathfrak{h}_0^*} \dim(\operatorname{Ker} \hat{\gamma}) \leq n + \min_{\gamma \in \mathfrak{z}(e_1)^*} \dim(\operatorname{Ker} \hat{\gamma})$. Now, we make a special choice for γ. Set $\gamma(\xi_1^{1,d_1}) = 1$ and $\gamma(\xi_i^{j,s}) = 0$ for all other $\xi_i^{j,s}$. The subspace $\mathfrak{h}_1$ is

generated by the vectors $\xi_i^{1,s}$ and $\xi_1^{i,s}$ with $i \neq 1$. We have

$$\begin{cases} \hat{\gamma}(\xi_1^{i,s}, \xi_i^{1,d_1-s}) = 1, \\ \hat{\gamma}(\xi_1^{i,s}, \xi_i^{1,b}) = 0 \text{ if } s+b \neq d_1, \\ \hat{\gamma}(\xi_1^{i,s}, \xi_1^{i,b}) = \hat{\gamma}(\xi_i^{1,s}, \xi_i^{1,b}) = 0. \end{cases}$$

The form $\hat{\gamma}$ defines a non-degenerate pairing between the spaces $U_1^i := \langle \xi_1^{i,s} | 0 \le s \le d_i \rangle$ and $U_i^1 := \langle \xi_i^{1,s} | d_1 - d_i \le s \le d_1 \rangle$. Hence, $\hat{\gamma}$ is non-degenerate and $\operatorname{ind} \mathfrak{z}(e) \le n$.

3. Symplectic algebra

In this section $\mathfrak{g} = \mathfrak{sp}_{2n} = \mathfrak{sp}(V)$, where V in an $2n$-dimensional vector space over $\mathbb{K}$. As above, $e \in \mathfrak{sp}_{2n}$ is a nilpotent element and $\mathfrak{z}(e) \subset \mathfrak{g}$ is the centraliser of e. Let $\{w_i\}$ be generators of Jordan blocks associated with e. We may assume that the space of each even-dimensional Jordan block is orthogonal to the space of all other Jordan blocks. If d_i is even, then the restriction of $\mathfrak{sp}_{2n}$-invariant form $(\ ,\ )_V$ on the space of i-th Jordan block is zero. One can choose generators $\{w_i\}$ such that the odd-dimensional blocks are partitioned in pairs (i, i'), where i' is the number of the unique Jordan block which is not orthogonal to the i-th one. Note that $d_{i'} = d_i$.

Let $\mathfrak{z}_{\mathfrak{gl}}(e)$ be the centraliser of e in $\mathfrak{gl}_{2n}$. Recall that $\mathfrak{z}(e) = \mathfrak{z}_{\mathfrak{gl}}(e)^\sigma \oplus \mathfrak{z}_1$, where σ is an involutive automorphism of $\mathfrak{gl}_{2n}$. For elements of $\mathfrak{z}_{\mathfrak{gl}}(e)$ we use notation introduced in the previous section.

Let $\alpha \in \mathfrak{z}_{\mathfrak{gl}}(e)^*$ be a function determined just like in the previous case:

$$\alpha(\varphi) = a_1 \cdot c_1^{1,d_1} + a_2 \cdot c_2^{2,d_2} + \ldots + a_{2n} \cdot c_k^{k,d_k},$$

where φ is given by its coefficients $c_i^{j,s}$, and $\{a_i\}$ are pairwise distinct non-zero numbers with $a_{i'} = -a_i$.

Lemma 2. *In the above notation, we have $\alpha(\mathfrak{z}_1) = 0$.*

Proof. Assume that there is $\psi \in \mathfrak{z}_1$ such that $\alpha(\psi) \neq 0$. Then there is a non-zero coefficient c_i^{i,d_i} of ψ. Recall that $\sigma(\psi) = -\psi$. The element ψ multiplies the $\mathfrak{sp}_{2n}$-invariant skew-symmetric form $(\ ,\ )_V$ by -1, in particular, $(\psi(w_i), v)_V = (w_i, \psi(v))_V$ for each vector $v \in V$. Clearly, $\psi(w_i)$ and w_i have to be orthogonal with respect to the skew-symmetric form. If d_i is odd, then $(w_i, e^{d_i} w_i)_V \neq 0$, hence, $c_i^{i,d_i} = 0$. If on the contrary d_i is even, then

$$\begin{aligned} c_i^{i,d_i}(e^{d_i} w_i, w_{i'})_V = (\psi(w_i), w_{i'})_V = (w_i, \psi(w_{i'}))_V = c_{i'}^{i',d_i}(w_i, e^{d_i} w_{i'})_V = \\ = (-1)^{d_i} c_{i'}^{i',d_i}(e^{d_i} w_i, w_{i'})_V = c_{i'}^{i',d_i}(e^{d_i} w_i, w_{i'})_V. \end{aligned}$$

Hence, $c_i^{i,d_i} = c_{i'}^{i',d_i}$. Combining this equality with defining formula of α we get a sum over pairs of odd-dimensional blocks

$$\alpha(\psi) = \sum_{(i,i')} (a_i + a_{i'}) c_i^{i,d_i},$$

which is zero since $a_i = -a_{i'}$. □

Denote by $\tilde{\alpha}$ the restriction of α to $\mathfrak{z}(e)$.

Theorem 2. *The dimension of the stabiliser $\mathfrak{z}(e)_{\tilde\alpha} = \mathfrak{z}_{\mathfrak{gl}}(e)_\alpha \cap \mathfrak{sp}_{2n}$ equals n.*

Proof. The stabiliser of α in $\mathfrak{z}_{\mathfrak{gl}}(e)$ consist of all maps preserving the spaces of the Jordan blocks. By Proposition 1, $\mathfrak{z}(e)_{\tilde\alpha} = \mathfrak{z}_{\mathfrak{gl}}(e)_\alpha \cap \mathfrak{z}(e)$. Describe the intersection of $\mathfrak{z}_{\mathfrak{gl}}(e)_\alpha$ with the symplectic subalgebra. If w_i is a generator of an even-dimensional block, then $\xi_i^{i,s}$ multiply the skew-symmetric form by $(-1)^{s+1}$, i.e., $(\xi_i^{i,s}(e^b w_i), e^t w_i) = (-1)^s (e^b w_i, \xi_i^{i,s}(e^t w_i))$. Consider a space of a pair (i,i') of odd-dimensional blocks. Set $d := d_i = d_{i'}$. Recall that $(w_i, e^d w_{i'}) = (-1)^s (e^s w_i, e^{d-s} w_{i'}) = -(w_{i'}, e^d w_i)$. Since $(e^s w_i, e^{d-s} w_{i'}) = (-1)^s (w_i, e^d w_{i'})$, the elements $\xi_i^{i,s} + (-1)^{s+1} \xi_{i'}^{i',s}$ preserve the skew-symmetric form, and the elements $\xi_i^{i,s} + (-1)^s \xi_{i'}^{i',s}$ multiply it by -1. From each even-dimensional block i we get $(d_i+1)/2$ vectors, and from a pair (i,i') we get $d_i + 1$ vectors. Thus the stabiliser of α in the whole of $\mathfrak{sp}_{2n}$ is an n-dimensional subalgebra. $\square$

4. The orthogonal case

In this section $\mathfrak{g} = \mathfrak{so}_n$. As above $e \in \mathfrak{so}_n$ is a nilpotent element, $\mathfrak{z}(e)$ is the centraliser of e in $\mathfrak{g}$. Let $\{w_i\}$ be generators of Jordan blocks associated with e. We may assume that the space of each odd-dimensional Jordan block is orthogonal to the space of all other Jordan blocks. If d_i is odd, then the restriction of $\mathfrak{so}_n$-invariant form $(\ ,\ )_V$ on the space of i-th Jordan block is zero. One can choose generators $\{w_i\}$ such that the even-dimensional blocks are partitioned in pairs (i, i^*), where i^* is the number of the unique Jordan block which is not orthogonal to the i-th one. Note that $d_{i^*} = d_i$.

Like the symplectic algebra, the orthogonal algebra is a symmetric subalgebra of $\mathfrak{gl}_n$. Denote by σ the involution defining it. Since $\sigma(e) = e$, we have $\mathfrak{z}_{\mathfrak{gl}}(e) = \mathfrak{z}(e) \oplus \mathfrak{z}_1$ similarly to the symplectic case. If d_i is even, set $i^* = i$. Assume that $(w_{i*}, e^{d_i} w_i)_V = \pm 1$ and $(w_i, e^{d_i} w_i)_V = 1$ for $i = i^*$. The algebra $\mathfrak{z}(e)$ is generated (as a vector space) by the vectors $\xi_i^{j,d_j-s} + \varepsilon(i,j,s)\xi_{j^*}^{i^*,d_i-s}$, where $\varepsilon(i,j,s) = \pm 1$ depending on i,j and s. In its turn, the subspace $\mathfrak{z}_1$ is generated by the vectors $\xi_i^{j,d_j-s} - \varepsilon(i,j,s)\xi_{j^*}^{i^*,d_i-s}$. Recall that $(e^s w_i, e^{d_i-s} w_{i^*})_V \neq 0$ if $e^s w_i \neq 0$.

We give some simple examples of linear functions with zero restrictions to $\mathfrak{z}_1$. Let $\varphi \in \mathfrak{z}_{\mathfrak{gl}}(e)$ be a linear map defined by Formula (1). Set $\beta_i(\varphi) = c_i^{i,d_i-1}$, $\gamma_{i,j}(\varphi) = c_i^{j,d_j}$.

Lemma 3. *If $i = i^*$, $j = j^*$, $t \neq t^*$, then functions β_i, $\gamma_{i,j} - \gamma_{j,i}$ and $\gamma_{t,t} + \gamma_{t^*,t^*}$ are equal to zero on $\mathfrak{z}_1$.*

Proof. Suppose $\psi \in \mathfrak{z}_1$ is defined by Formula (1). Since $\sigma(\psi) = -\psi$ and $(\psi(w_i), ew_i)_V = c_i^{i,d_i-1}(e^{d_1-1}w_i, ew_i)_V$, we have

$$(\psi(w_i), ew_i)_V = (w_i, \psi(ew_i))_V = (w_i, e\psi(w_i))_V = -(ew_i, \psi(w_i))_V = -c_i^{i,d_i-1}(ew_i, e^{d_1-1}w_i)_V.$$

The form $(\ ,\ )_V$ is symmetric and $(ew_i, e^{d_i-1}w_i)_V \neq 0$, hence $\beta_i(\psi) = c_i^{i,d_i-1} = 0$.

Similarly,

$$c_i^{j,d_j}(e^{d_j}w_j, w_j)_V = (\psi(w_i), w_j)_V = (w_i, \psi(w_j))_V = c_j^{i,d_i}(w_i, e^{d_i}w_i)_V;$$
$$c_t^{d_t,t}(e^{d_t}w_t, w_{t*})_V = (\psi(w_t), w_{t^*})_V = (w_t, \psi(w_{t^*}))_V = c_{t^*}^{t^*,d_t}(w_t, e^{d_t}w_{t^*})_V.$$

Recall that by our choice $(e^{d_j}w_j, w_j)_V = (w_i, e^{d_i}w_i)_V = 1$, $(e^{d_t}w_t, w_{t*})_V = -(w_t, e^{d_t}w_{t^*})_V$. Hence $c_i^{j,d_j} = c_j^{i,d_i}$, $c_t^{t,d_t} = -c_{t^*}^{t^*,d_t}$. □

Let us prove the inequality $\operatorname{ind}\mathfrak{z}(e) \le \operatorname{rk}\mathfrak{so}_n$ by the induction on n. In the following two cases, the induction argument does not go through. Therefore we consider them separately. *The first case.* If $e \in \mathfrak{so}_{2m+1}$ is a regular nilpotent element, then $\mathfrak{z}(e)$ is a commutative m-dimensional algebra.
The second case. Let $e \in \mathfrak{so}_{4d}$ be a nilpotent element with two Jordan blocks of size $2d$ each. Set $\alpha(\varphi) = c_1^{1,2d-2} - c_2^{2,2d-2}$, where φ is defined by Formula (1). One can easily check that $\mathfrak{z}(e)_\alpha$ has a basis $\xi_1^{1,s} + (-1)^{s+1}\xi_2^{2,s}$ with $0 \le s \le 2d-1$ and that $\dim\mathfrak{z}(e)_\alpha = 2d$.

Order the Jordan blocks of e according to their dimensions $d_1 \ge d_2 \ge \ldots \ge d_k$. Here $d_i + 1$ stands for the dimension of the i-th Jordan block, similarly to the case of $\mathfrak{gl}_n$. Note that the numbers n and k have the same parity. Assume that $k > 1$ and if $k = 2$, then both Jordan blocks are odd dimensional. Then we have the following three possibilities:

(1) for some even number $2p < k$ the restriction of $(\ ,\ )_V$ to the space of the first $2p$ Jordan blocks is non-degenerate;

(2) the number d_i is even for $i = 1, k$ and odd for all other i;

(3) the number d_i is even if and only if $i = 1$.

Each of these three possibilities is considered separately. In the first two cases we make an induction step. In the third one a point $\alpha \in \mathfrak{z}(e)^*$ is given such that $\dim\mathfrak{z}(e)_\alpha \le \operatorname{rk}\mathfrak{so}_n$.
(1) Suppose the space V_{2m} of the first $2p$ Jordan blocks has dimension $2m$ and the restriction of $(\ ,\ )_V$ to V_{2m} is non-degenerate. Then $V = V_{2m} \oplus V_{\text{oth}}$, $e = e_1 + e_2$, where $e_1 \in \mathfrak{so}_{2m}$, $e_2 \in \mathfrak{so}_{n-2m}$. Let τ be an involution of $\mathfrak{gl}_n$ corresponding to these direct sum, i.e., $\mathfrak{gl}_n^\tau = \mathfrak{gl}(V_{2m}) \oplus \mathfrak{gl}(V_{\text{oth}})$. Set $\mathfrak{h} = \mathfrak{z}(e)$, $\mathfrak{h}_0 = \mathfrak{z}(e)^\tau$. Then $\mathfrak{h}_0 = \mathfrak{z}(e_1) \oplus \mathfrak{z}(e_2)$, where the centralisers of e_1 and e_2 are taken in $\mathfrak{so}_{2m}$ and $\mathfrak{so}_{n-2m}$, respectively. By the inductive hypothesis, $\operatorname{ind}\mathfrak{z}(e_1) = m$, $\operatorname{ind}\mathfrak{z}(e_2) = [n/2] - m$. Hence, $\operatorname{ind}\mathfrak{z}(e) \le [n/2] + \min\limits_{\gamma\in\mathfrak{h}_0^*}\dim(\operatorname{Ker}\hat\gamma)$. To conclude we have to point out a function $\gamma \in \mathfrak{h}_0^*$ such that $\hat\gamma$ is non-degenerate. Recall that the involutions σ and τ commute with each other, preserve e and determine the decomposition $\mathfrak{z}_{\mathfrak{gl}}(e) = (\mathfrak{z}(e_1) \oplus \mathfrak{z}(e_2) \oplus \mathfrak{h}_1) \oplus \mathfrak{z}_1$. If $\gamma(\mathfrak{z}_1) = \gamma(\mathfrak{h}_1) = 0$, then $\operatorname{Ker}\hat\gamma = (\mathfrak{h}_1 \cap \mathfrak{z}_{\mathfrak{gl}}(e)_\gamma)$.

Divide odd-dimensional Jordan blocks into pairs (i, i') (it is assumed that $i, i' \le 2p$). Define a point γ by

$$\gamma(\varphi) = \sum_{(i,i'),\, i,i'\le 2p} (c_i^{i',d_{i'}} - c_{i'}^{i,d_i}) + \sum_{j\le 2p,\,(d_j+1)\text{ is even}} c_j^{j,d_j},$$

where $\varphi \in \mathfrak{z}_{\mathfrak{gl}}(e)$ is given by its coefficients $c_i^{j,s}$. The first summand is a sum of $(\gamma_{i,i'} - \gamma_{i',i})$ over pairs of odd-dimensional blocks, the second is the sum of $(\gamma_{j,j} + \gamma_{j^*,j^*})$ over pairs of even-dimensional blocks. According to Lemma 3, both summands are identical zeros on $\mathfrak{z}_1$. Moreover, by the definition $\gamma(\mathfrak{h}_1) = 0$.

Set $j' := j$ for even-dimensional blocks. Assume that an element $\psi \in \mathfrak{h}_1$ determined by (1) lies in the kernel of $\hat\gamma$, i.e., $\gamma([\psi, \mathfrak{h}_1]) = 0$. Then $\gamma([\psi, \mathfrak{h}]) = \gamma([\psi, \mathfrak{z}_{\mathfrak{gl}}(e)]) = 0$. Since $\psi \in \mathfrak{so}_n$

and $\psi \neq 0$, we may assume that $c_i^{j,s} \neq 0$ for some $j > 2p \geq i$. We have

$$\gamma([\psi, \eta_i^{j',d_{j'}}]) = \pm c_i^{j,s} \neq 0.$$

Thus we have proved that $\hat{\gamma}$ is non-degenerate and $\operatorname{ind} \mathfrak{z}(e) \leq [n/2]$.

(2) Consider a decomposition $V = V_{\mathrm{oth}} \oplus V_{d_k+1}$, where the second summand is the space of the smallest (odd-dimensional) Jordan block and the first one is the space of all other blocks. As above $e = e_1 + e_2$, where nilpotent element e_2 corresponds to the smallest (odd-dimensional) Jordan block. We define an involution τ, algebras $\mathfrak{z}(e_1)$, $\mathfrak{z}(e_2)$, $\mathfrak{h}_0$ and a subspace $\mathfrak{h}_1$ in the same way as in case **(1)**. By the inductive hypothesis $\operatorname{ind} \mathfrak{z}(e_1) = [(n - d_k - 1)/2]$, $\operatorname{ind} \mathfrak{z}(e_2) = d_k/2$. Hence $\operatorname{ind} \mathfrak{h}_0 = n/2 - 1$. By Lemma 1, $\operatorname{ind} \mathfrak{z}(e) \leq \operatorname{ind} \mathfrak{h}_0 + \min_{\gamma \in \mathfrak{h}_0^*} \dim(\operatorname{Ker} \hat{\gamma})$. Let γ be the following function

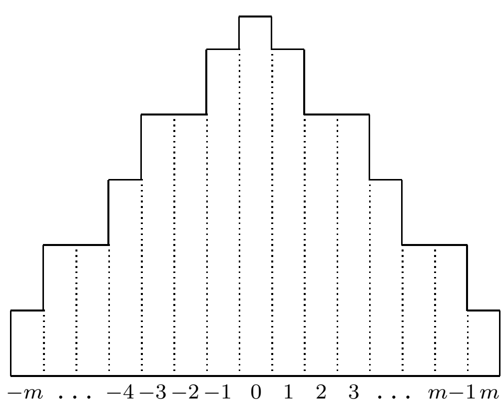
$$\gamma(\varphi) = c_1^{1,d_1-1} + \sum_{i=2}^{k-1} c_i^{i,d_i},$$

where φ is given by formula (1). The first summand is β_1, the second summand is a sum of $(\gamma_{j,j} + \gamma_{j^*,j^*})$ over pairs of even-dimensional blocks. Due to Lemma 3, $\gamma(\mathfrak{z}_1) = 0$. Suppose $\psi \in \mathfrak{h}_1$ is given by its coefficients $c_i^{j,s}$. Then

$$\begin{cases} \gamma([\xi_1^{k,b}, \psi]) = c_k^{1,d_1-1-b}, \\ \gamma([\xi_i^{k,b}, \psi]) = c_k^{i,d_i-b} \quad \text{for } 1 < i < k. \end{cases}$$

One can see that the kernel of $\hat{\gamma}$ is one-dimensional and generated by $(\xi_k^{1,d_1} - \xi_1^{k,d_k})$. Hence, $\operatorname{ind} \mathfrak{z}(e) \leq [n/2] - 1 + 1 = [n/2]$.

(3) In this case n and k are odd, and e has a unique odd-dimensional Jordan block whose size is maximal. Assume that $k = 2m + 1$. Enumerate the Jordan blocks by integers ranging from $-m$ to m. Let the unique odd-dimensional block has number zero. Suppose that pairs of blocks $(-i, i)$ and $(-j, j)$ are orthogonal to each other if $i \neq \pm j$, and dimensions of Jordan blocks are increasing from $-m$ to 0 and decreasing from 0 to m, i.e., if $|i| \leq |j|$, then $d_i \geq d_j$. Note that $d_i = d_{-i}$. Such enumeration is shown on Picture 1. Choose the generators w_i of Jordan blocks such that $i(w_i, e^{d_i} w_{-i})_V = |i|$ for $i \neq 0$ and $(w_0, e^{d_0} w_0)_V = 1$.



Picture 1.

Suppose $\varphi \in \mathfrak{z}_{\mathfrak{gl}}(e)$ is given by Formula (1). Consider the following point $\alpha \in \mathfrak{z}_{\mathfrak{gl}}(e)^*$:

$$\alpha(\varphi) = \sum_{i=-m+1}^{m} c_{i-1}^{i,d_i}.$$

One can check by direct computation that $c_{i-1}^{i,d_i}(\psi) = -c_{-i}^{1-i,d_{1-i}}(\psi)$ for each $\psi \in \mathfrak{z}_1$ and, hence, $\alpha(\mathfrak{z}_1) = 0$. Let $\tilde{\alpha} \in \mathfrak{z}(e)^*$ be the restriction of α. Let us describe the stabiliser $\mathfrak{z}(e)_{\tilde{\alpha}} = \mathfrak{z}_{\mathfrak{gl}}(e)_\alpha \cap \mathfrak{z}(e)$. Note that $\alpha([\varphi, \xi_i^{j,s}]) = c_{j-1}^{i,d_j-s}(\varphi) - c_j^{i+1,d_{i+1}-s}(\varphi)$.

Lemma 4. *Suppose $\varphi \in \mathfrak{z}(e)$ and $\mathrm{ad}^*(\varphi)\alpha = 0$. Then $c_i^{j,s} = c_i^{j,s}(\varphi) = 0$ for $i < j$.*

Proof. Assume that the statement is wrong and take a maximal i for which there are $j > i$ and s such that $c_i^{j,s} \neq 0$. Because φ preserves $(\ ,\ )_V$, $c_{-j}^{-i,d_i-d_j+s} = \pm c_i^{j,s} \neq 0$. Hence, $-j \le i < j$, $j > 0$, $|i| \le j$ and $d_i \ge d_j$. Moreover, $-j < (i+1) \le j$ and $d_{i+1} \ge d_j$. Evidently, $d_{i+1} - s \ge d_j - s \ge 0$ and there is an element $\xi_j^{i+1,d_{i+1}-s} \in \mathfrak{z}_{\mathfrak{gl}}(e)$. We have

$$0 = \alpha([\varphi, \xi_j^{i+1,d_{i+1}-s}]) = c_i^{j,s} - c_{i+1}^{j+1,\delta} = c_i^{j,s}.$$

Here we do not give a precise value of δ. Anyway all coefficients $c_{i+1}^{j+1,b}$ are zeros, because $j+1 > i+1 > i$. We get a contradiction. Thus the lemma is proved. $\square$

Let us say that $\varphi \in \mathfrak{z}_{\mathfrak{gl}}(e)$ has a *step* l whenever $c_i^{j,s}(\varphi) = 0$ for $j \neq i + l$. Each vector $\varphi \in \mathfrak{z}_{\mathfrak{gl}}(e)$ can be represented as a sum $\varphi = \varphi_{-2m} + \varphi_{-2m+1} + \ldots + \varphi_{2m-1} + \varphi_{2m}$, where the step of φ_l equals l. The notion of the step is well-defined on $\mathfrak{z}(e)$, due to an equality $(-i) - (-j) = j - i$. From the definition of α, one can deduce that $\alpha(\varphi_l, \varphi_t) \neq 0$ only if $l + t = 1$. The stabiliser $\mathfrak{z}(e)_\alpha$ is a direct sum of its subspaces Φ_l, consisting of elements having step l. As we have seen, $\Phi_l = \varnothing$ if $l > 0$. It is remains to describe elements with non-positive steps.

Example 2. Let us show that $\dim \Phi_0 \le d_0/2$. Suppose $\varphi \in \Phi_0$, $\varphi \neq 0$ and $\varphi(w_0) = 0$. Take a minimal by the absolute value i such that $\varphi(w_i) \neq 0$. Since $\varphi \in \mathfrak{so}_n$, we have also $\varphi(w_{-i}) \neq 0$. Assume that $i > 0$ and a coefficient $c_i^{i,s}$ of φ is non-zero. Then $|i-1| < i$, $d_{i-1} \ge d_i$, there is an element $\xi_{i-1}^{i,d_i-s} \in \mathfrak{z}_{\mathfrak{gl}}(e)$ and $0 = \alpha([\xi, \varphi]) = c_i^{i,s} - c_{i-1}^{i-1,s} = c_i^{i,s}$. Hence, if $\varphi(w_0) = 0$, then also $\varphi = 0$. Thus, a vector $\varphi \in \Phi_0$ is entirely determined by its value on w_0. In its turn $\varphi(w_0) = c_1 e w_0 + c_3 e^3 w_0 + \ldots + c_{d_0-1} e^{d_0-1} w_0$.

Lemma 5. *If $q = 2l$ or $q = 2l-1$, where $0 < l \le m$, then $\dim \Phi_{-q} \le (d_l+1)/2$.*

Proof. Similarly to the previous example, we show that if $\varphi \in \Phi_{-q}$ and $\varphi(w_l) = 0$, then also $\varphi = 0$. Since $\varphi \in \mathfrak{so}_n$, if $\varphi(w_i) \neq 0$, then also $\varphi(w_{q-i}) \neq 0$. Suppose $\varphi(w_j) \neq 0$ for some j. If $j < l$, then $j - q \ge l$, but $\varphi(w_l) = 0$, hence $j > l$. Find the minimal $j > l$ such that $\varphi(w_j) \neq 0$. Suppose $c_j^{j-q,s} = c_j^{j-q,s}(\varphi) \neq 0$. We have $-j < -l \le j - q - 1 < j$, $d_j \le d_{j-q-1}$, $d_{j-q} - s \ge 0$. Hence, there is an element $\xi := \xi_{j-q-1}^{j,d_{j-q}-s} \in \mathfrak{z}_{\mathfrak{gl}}(e)$. As above $0 = \alpha([\xi, \varphi,]) = c_j^{j-q,s} - c_{j-1}^{j-q-1,\delta} = c_j^{j-q,s}$ (we do not give a precise value of δ, anyway, $\varphi(w_{j-1}) = 0$, since $l \le j-1 < j$). To conclude we describe possible values $\varphi(w_l)$. If $q = 2l$, then $\varphi(w_l) = c_0 w_{-l} + c_2 e^2 w_{-l} + \ldots + c_{d_l} e^{d_l} w_{-l}$. In case $q = 2l-1$ we get an equation on

coefficients of φ: $0 = \alpha[(\xi^{l,b}_{-l}, \varphi]) = c_l^{-l+1,d_{l-1}-b} - c_{l-1}^{-l,d_l-b}$, i.e., $c_l^{-l+1,d_{l-1}-b} = c_{l-1}^{-l,d_l-b}$. This is possible only for odd b. □

Theorem 3. *Suppose $e \in \mathfrak{so}_n$ is a nilpotent element. Then* $\operatorname{ind}\mathfrak{z}(e) = \operatorname{rk}\mathfrak{so}_n = [n/2]$.

Proof. If possibility **(3)** takes place, i.e., only one Jordan block of e is odd-dimensional and it is also maximal, then, as we have seen, $\mathfrak{z}(e)_{\tilde\alpha} = \bigoplus\limits_{q=0}^{2m} \Phi_{-q}$. Moreover, $\dim \Phi_q$ is at most half of dimension of the Jordan block with number $[(q+1)/2]$. Thereby, $\dim \mathfrak{z}(e)_{\tilde\alpha} \le ((d_0+1)/2 + (\sum\limits_{l=1}^{m} d_l)) = [n/2]$. On the other hand, according to Vinberg's inequality, $\operatorname{ind}\mathfrak{z}(e) \ge [n/2]$.

In cases **(1)** and **(2)** the inequality $\operatorname{ind}\mathfrak{z}(e) \le \operatorname{rk}\mathfrak{so}_n$ was proved by induction.

If none of these three possibilities takes place, then either $k = 1$ and e is a regular nilpotent element, or $k = 2$ and both Jordan blocks of e are even-dimensional. These two cases have been considered separately. □

5. Generic points

In this section we assume that $\mathbb{K}$ is algebraically closed and of characteristic zero. Suppose we have a linear action of a Lie algebra $\mathfrak{g}$ on a vector space V.

Definition. A vector $x \in V$ (a subalgebra $\mathfrak{g}_x$) is called *a generic point* (*a generic stabiliser*), if for every point $y \in U \subset V$ of some open in Zariski topology subset U algebras $\mathfrak{g}_y$ and $\mathfrak{g}_x$ are conjugated in $\mathfrak{g}$.

It is well known that generic points exist for any linear action of a reductive Lie algebra.

It is proved in [5, §1] that a subalgebra $\mathfrak{g}_x$ is a generic stabiliser if and only if $V = V^{\mathfrak{g}_x} + \mathfrak{g}x$, where $V^{\mathfrak{g}_x}$ is the subspace of all vectors of V invariant under $\mathfrak{g}_x$.

Tauvel and Yu have noticed that in case of a coadjoint representation $\mathfrak{g}x = (\mathfrak{g}/\mathfrak{g}_x)^* = \operatorname{Ann}(\mathfrak{g}_x)$, $(\mathfrak{g}^*)^{\mathfrak{g}_x} = \operatorname{Ann}([\mathfrak{g}_x, \mathfrak{g}])$. From this observation they have deduced a simple and useful criterion.

Theorem 4. [13, Corollaire 1.8.] *Let $\mathfrak{g}$ be a Lie algebra and $x \in \mathfrak{g}^*$. The subalgebra $\mathfrak{g}_x$ is a generic stabiliser of the coadjoint representation of $\mathfrak{g}$ if and only if $[\mathfrak{g}_x, \mathfrak{g}] \cap \mathfrak{g}_x = \{0\}$.*

Unfortunately, the authors of [13] were not aware of the aforementioned Élashvili's result and have proved it anew.

Let $e \in \mathfrak{gl}_n$ be a nilpotent element and $\mathfrak{z}(e)$ the centraliser of e. Set $\mathfrak{h} = \mathfrak{z}(e)_\alpha$, where $\alpha \in \mathfrak{z}(e)^*$ is the same as in Section 2.

Proposition 2. *There is an $\mathfrak{h}$-invariant decomposition $\mathfrak{z}(e) = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{m}$ is generated by the vectors $\xi_i^{j,s}$ with $i \ne j$.*

Proof. Recall that $\mathfrak{h}$ is generated by the vectors $\xi_i^{i,s}$. The inclusion $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ follows immediately from the equality

$$[\xi_i^{i,s}, \xi_j^{t,b}] = \begin{cases} \xi_i^{t,s+b} & \text{if } i = j, i \ne t; \\ -\xi_j^{i,s+b} & \text{if } i = t, i \ne j; \\ 0 & \text{otherwise.} \end{cases}$$

$\square$

There is a similar decomposition in the case of symplectic algebras. Let $e \in \mathfrak{sp}(V) \subset \mathfrak{gl}(V)$. Denote by $\mathfrak{z}_{\mathfrak{gl}}(e)$ and $\mathfrak{z}_{\mathfrak{sp}}(e)$ the centralisers of e in $\mathfrak{gl}(V)$ and $\mathfrak{sp}(V)$, respectively. We use notation of Section 3. Suppose $\mathfrak{z}_{\mathfrak{gl}}(e) = \mathfrak{h} \oplus \mathfrak{m}$. Evidently, this decomposition is σ-invariant and $\mathfrak{z}_{\mathfrak{sp}}(e) = \mathfrak{h}^\sigma \oplus \mathfrak{m}^\sigma$, where $\mathfrak{h}^\sigma = \mathfrak{z}_{\mathfrak{sp}}(e)_{\tilde\alpha}$.

Theorem 5. *The Lie algebras $\mathfrak{z}_{\mathfrak{gl}}(e)_\alpha$ and $\mathfrak{z}_{\mathfrak{sp}}(e)_{\tilde\alpha}$ constructed in Sections 2 and 3 in cases of general linear and symplectic algebras are generic stabilisers of the coadjoint actions of $\mathfrak{z}_{\mathfrak{gl}}(e)$ and $\mathfrak{z}_{\mathfrak{sp}}(e)$.*

Proof. Let us verify the condition of Theorem 4. Since $[\mathfrak{h}, \mathfrak{z}_{\mathfrak{gl}}(e)] = [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$, we have $[\mathfrak{h}, \mathfrak{z}_{\mathfrak{gl}}(e)] \cap \mathfrak{h} = 0$. Similarly, $[\mathfrak{h}^\sigma, \mathfrak{z}_{\mathfrak{sp}}(e)] \subset \mathfrak{m}^\sigma$. $\square$

In case of orthogonal algebras it can happen that a generic stabiliser of the coadjoint action of $\mathfrak{z}(e)$ does not exist.

Example 3. Let $e \in \mathfrak{so}_8$ be a subregular nilpotent element. Then it has two Jordan blocks of dimensions 3 and 5. Choose the generators w_1, w_2 of Jordan blocks such that $(w_1, e^2 w_1)_V = (w_2, e^4 w_2)_V = 1$. The dimension of $\mathfrak{z}(e)$ is 6 and $\mathfrak{z}(e)$ has a three-dimensional center, generated by the vectors e, $e^3 = \xi_2^{2,3}$ and $\varphi_3 = \xi_1^{2,4} - \xi_2^{1,2}$. Since $\operatorname{ind} \mathfrak{z}(e) = 4$, we have $\dim \mathfrak{z}(e)_\alpha = 4$ for points of some open subset $U \subset \mathfrak{z}(e)^*$.

Assume that a generic stabiliser of the coadjoint action of $\mathfrak{z}(e)$ exists and denote it by $\mathfrak{f}$. Evidently, $\mathfrak{f}$ contains the center of $\mathfrak{z}(e)$. Consider an element $\varphi_2 = \xi_1^{2,3} + \xi_2^{1,1} \in \mathfrak{z}(e)$. Clearly, the subspace $[\varphi_2, \mathfrak{z}(e)]$ is a linear span of e^3 and φ_3. In particular, it is contained in the center of $\mathfrak{z}(e)$, and, hence, in $\mathfrak{f}$. Hence, $[\varphi_2, \mathfrak{f}] \subset \mathfrak{f}$, and, by Theorem 4, $\mathfrak{f} \subset \mathfrak{z}(e)_{\varphi_2}$. Since $\dim \mathfrak{z}(e)_{\varphi_2} = 4$, we have $\mathfrak{f} = \mathfrak{z}(e)_{\varphi_2}$. On the other hand, $\mathfrak{z}(e)_{\varphi_2} = \langle e, e^3, \varphi_3, \varphi_2 \rangle_{\mathbb{K}}$ is a normal, but not a central subalgebra of $\mathfrak{z}(e)$.

Consider the embedding $\mathfrak{so}_8 \subset \mathfrak{so}_9$ as the stabiliser of the first basis vector in $\mathbb{K}^9$. By a similar argument one can show that a generic stabiliser does not exist for the coadjoint action of $\mathfrak{z}_{\mathfrak{so}_9}(e)$ either.

6. Commuting varieties

Let $\mathfrak{g}$ be a Lie algebra over an algebraically closed field $\mathbb{K}$ of characteristic zero. A closed subset $Y = \{(x, y) | x, y \in \mathfrak{g}, [x, y] = 0\} \subset (\mathfrak{g} \times \mathfrak{g})$ is called the *commuting variety* of the algebra $\mathfrak{g}$. The question of whether Y is irreducible or not is of a great interest. In case of a reductive algebra $\mathfrak{g}$ the commuting variety Y is irreducible and coincides with the closure of $G(\mathfrak{a}, \mathfrak{a})$, where $\mathfrak{a} \subset \mathfrak{g}$ is a Cartan subalgebra and G is a connected algebraic group with $\operatorname{Lie} G = \mathfrak{g}$.

Let $e \in \mathfrak{gl}_n$ be a nilpotent element and $\mathfrak{z}(e)$ the centraliser of e. We use notation introduced in Section 2. Set $\mathfrak{h} = \mathfrak{z}(e)_\alpha$. Consider a subalgebra $\mathfrak{t} \subset \mathfrak{z}(e)$ generated by the vectors $\xi_i^{i,0}$. Evidently, $\mathfrak{t} \subset \mathfrak{h}$. Moreover, since $[\xi_i^{j,s}, t_i \xi_i^{i,0} + t_j \xi_j^{j,0}] = (t_j - t_i) \xi_i^{j,s}$, the algebra $\mathfrak{h}$ coincides with the normaliser (= centraliser) of $\mathfrak{t}$ in $\mathfrak{z}(e)$. Hence, $\mathfrak{h}$ coincides with its normaliser in $\mathfrak{z}(e)$.

Let $Z(e)$ be the identity component of the centraliser of e in GL_n. Then $Y_0 = \overline{Z(e)(\mathfrak{h},\mathfrak{h})}$ is an irreducible component of Y of maximal dimension. As in the reductive case, Y is irreducible if and only if $Y_0 = Y$. It is known that if a nilpotent element e has at most two Jordan blocks, then Y is irreducible [7]. In the general case, the statement is not true, since it would lead to the irreducibility of the commuting varieties of triples of matrices.

Example 4. Assume that $Y_0 = Y$ for all nilpotent elements $e \in \mathfrak{gl}_m$ with $m \le n$. Consider the set of triples of commuting matrices

$$C_3 = \{(A,B,C)|A,B,C \in \mathfrak{gl}_n, [A,B]=[A,C]=[B,C]=0\}.$$

Let $\mathfrak{a} \subset \mathfrak{gl}_n$ be a subalgebra of diagonal matrices. Clearly, $\overline{\mathrm{GL}_n(\mathfrak{a},\mathfrak{a},\mathfrak{a})}$ is an irreducible component of C_3. Let us prove by induction that it coincides with C_3. There is nothing to prove for $n=1$. Let $n>1$. We show that each triple (A,B,C) of commuting matrices is contained in the closure $\overline{\mathrm{GL}_n(\mathfrak{a},\mathfrak{a},\mathfrak{a})}$. Without loss of generality, we may assume that $A,B,C \in \mathfrak{sl}_n$. Let $A = A_s + A_n$ be the Jordan decomposition of A. If $A_s \ne 0$, consider the centraliser $\mathfrak{z}(A_s)$ of A_s in $\mathfrak{gl}_n$. Clearly, $A,B,C \in \mathfrak{z}(A_s)$ and $\mathfrak{z}(A_s)$ is a sum of several algebras $\mathfrak{gl}_{n_i}$ with strictly smaller dimension. We may assume that $\mathfrak{a} \subset \mathfrak{z}(A_s)$. Then, by the inductive hypothesis

$$(A,B,C) \in \overline{Z(A_s)(\mathfrak{a},\mathfrak{a},\mathfrak{a})} \subset \overline{\mathrm{GL}_n(\mathfrak{a},\mathfrak{a},\mathfrak{a})}.$$

Suppose now that all three elements A,B,C are nilpotent and at least one of them, say A, is not regular. Consider the centraliser $\mathfrak{z}(A) \subset \mathfrak{gl}_n$. We have assumed that $Y_0 = Y$, i.e., the pair (B,C) lies in the closure of $Z(A)(\mathfrak{h},\mathfrak{h})$. It will be enough to show that $(A,\mathfrak{h},\mathfrak{h}) \subset \overline{\mathrm{GL}_n(\mathfrak{a},\mathfrak{a},\mathfrak{a})}$. Let $x \in \mathfrak{t} \subset \mathfrak{h}$ be a non-central semisimple element. Then $A \in (\mathfrak{gl}_n)_x$ and $\mathfrak{h} \subset (\mathfrak{gl}_n)_x$. Once again we can make an induction step, passing to a subalgebra $(\mathfrak{gl}_n)_x$.

If all three elements A,B,C are regular nilpotent, then there is a non-trivial linear combination A' of them, which is non-regular. In particular, the triple (A,B,C) is equivalent under the action of GL_n to some other triple (A',B',C') of commuting nilpotent matrices.

It is known that for $n > 31$ the variety C_3 is reducible, see [6]. Hence, the commuting variety Y is certainly reducible for some nilpotent elements. It will be interesting to find minimal (in some sense) nilpotent elements for which Y is reducible and/or describe some classes of nilpotent elements for which Y is irreducible.

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